ARISTOTLE UNIVERSITY OF THESSALONIKI

A Study of Alcubierre's Warp Drive Solution in General Relativity

by

Stoikos Evangelos

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Abstract

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by Stoikos Evangelos

STOIKOS EVANGELOS.

In the following thesis we have tried to initially acquaint the reader with the formalism of 3+1 in General Relativity, which is crucial for the understanding of the way the Warp Drive is made to work. Continuing on to the next chapter the reader became aware of the Warp Drive metric, as proposed by Alcubierre, and how one spaceship may, by using the definitions and the dilation of space-time itself, cover an arbitrary great distance in measured coordinate time, which in turn can become as small as an observer would like. Although the Warp Drive as an idea is very useful and ground breaking, when it is put to the test of Quantum Mechanics, in the form of the Quantum Inequality, and the energy conditions some very disturbing results occur in the case of exceedingly large negative! energy requirements and in the manner of very small Warp Bubble wall thickness and great travelling velocities. Finally it will be shown that with a minor modification of the original Warp Drive metric the still negative energy requirements will be reduced from the mass in the entire visible universe to just some solar masses order, which may still be absurdly large but it gives hope for future research.

Acknowledgements

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Chapter 1

3+1 Formalism

In this initial chapter we will try to deepen further into the 3+1 formalism of [GR]. This will be achieved by following some really explanatory notes from a summer school, which took place in Catania,Italy in 2011. Firstly the covariant expression of Einstein's field equation will be used. In this expression time and space are treated on equal grounds. The $3+1 formalism$ will be used a lot in the coming chapters, because it is a very useful way to decompose the metric tensor of space-time in its 3 spatial parts and its 1 time part taking a more compact and helpful form.

1.1 Eulerian Observers

First of all, the definition of the so called **Eulerian Observer** will be proven in this chapter and the same definition will be used many times in the coming chapters so its best to clarify the properties of these observers. If an arbitrary *time coordinate* t is chosen one needs to find the normal $\nabla_a t$ to the space-like hypersurfaces Σ_t. It is clearly known from the definition that the normal vector we are trying to find is *timelike* and if the future is chosen, as it is almost always done, as the direction where the *time coordinate* t increases the *future-pointing unit normal* can be defined to each slice of the space-like hypersurface Σ_t as follows:

$$n^\alpha = \frac{\nabla^\alpha t}{\sqrt{-\nabla_\beta t \nabla^\beta t}} \tag{1.1}$$

Also it is necessary for n^α to satisfy:

$$n_\alpha n^\alpha = -1 \tag{1.2}$$

Thus now n^α is defined as the 4-velocity of an observer, whose world-line is always normal to the hypersurface Σ_t, the so called **Eulerian Observer**. Another important definition that must be described for future references is the definition of the *projection tensor* onto the space-like hypersurfaces Σ_t, which is given by:

$$\gamma^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta \tag{1.3}$$

1.2 Vectors and the induced metric of 3+1 Space-Time

Considering the next step, before the metric element can finally be casted for the space-time in its $3+1$ form, some important objects must be defined. Initially, the vector t^α needs to be defined. It will be as shown from the figure in a general different direction than n^α, and decomposed such as:

$$t^\alpha = an^\alpha + \beta^\alpha \tag{1.4}$$

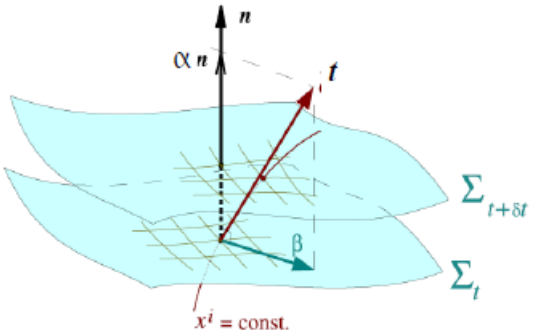

FIGURE 1.1: Arbitrary foliations of space-time

Next if the simple definition is applied, which states that the projection of the time-like vector along the unit normal is :

$$\begin{aligned} t^{\alpha} n_{\alpha} &= -a \\ \rightarrow a &= (-\nabla_{\alpha} t \nabla^{\alpha} t)^{-1/2} \end{aligned} \tag{1.5}$$

One comes to the conclusion of the useful lapse function a. Furthermore the following properties can be taken by definition:

$$\{n_{\alpha}\} = \{-\alpha \nabla_{\alpha} t\} = \{-\alpha, 0, 0, 0\} \tag{1.6}$$

Also

$$\{n^{\alpha}\} = \left\{ \frac{t^{\alpha} - \beta^{\alpha}}{\alpha} \right\} = \left\{ \frac{1}{\alpha}, \frac{-\beta^{i}}{\alpha} \right\} \tag{1.7}$$

So following these steps and knowing, as it was shown before, that n^{α} is the 4-velocity of the **Eulerian Observer** and using τ as the proper time of the observer and t as the coordinate time a relation can be derived for the *lapse function* :

$$\begin{aligned} n^{t} &= \frac{1}{\alpha} = \frac{dt}{d\tau} \\ \rightarrow d\tau &= \alpha dt \end{aligned} \tag{1.8}$$

Therefore, the *lapse function* is able to be defined as the function, which converts the universal time interval dt to the proper time interval $d\tau$ of the local **Eulerian Observer**. Lastly as it can be easily understood from Figure(1.1) the *lapse function* can be chosen freely as one can choose an arbitrary foliation of space-time.

Now for the final steps before the metric element can be cast in its new form the definitions of the *shift vector* and the *induced metric* onto Σ_t must be used. First the time vector t^{α} is being projected onto the hypersurface Σ_t and the definition of the *shift vector* is taken as:

$$\beta := \gamma^{\alpha}_{\beta} t^{\beta} \tag{1.9}$$

and its properties:

$$\begin{aligned} \beta^{\alpha} n_{\alpha} &= 0 \\ \rightarrow \{\beta^{\alpha}\} &= \left\{0, \beta^{i}\right\} \end{aligned} \tag{1.10}$$

.

As a final point, the metric tensor γ must also be projected, which will be named the *induced metric* onto Σ_t:

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta \tag{1.11}$$

Also it can be proven from its definition that this new metric can be used to raise and lower indices on the hypersurface Σ_t as follows:

$$\begin{aligned} &\gamma_{ij} = g_{ij} \\ &\gamma^{ik}\gamma_{kj} = \delta^i_j \\ &\rightarrow \gamma^{tt} = 0, \gamma^{ti} = 0 \end{aligned} \tag{1.12}$$

1.3 Metric Element in 3+1 Space-Time

To conclude with the final section, the previous definitions and results are going to be used and the metric element can now easily be cast for the space-time in its 3+1 form :

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{1.13}$$

Lastly, if one wants to find more precise informations about an **Eulerian Observer**, then by inserting $d\tau = \alpha dt$ in the metric element, one gets:

$$-d\tau^2 = ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{1.14}$$

This new equation now allows to interpret the *shift vector* as the relative velocity between the **Eulerian Observer** and the coordinate one.

$$\beta^i = -\frac{dx^i}{dt} \tag{1.15}$$

Chapter 2

Warp Drive Basics

In this next chapter with the new-found knowledge in [GR] and its 3+1 formalism the basic idea behind the Alcubierre **Warp Drive** will be described to the reader, proposed by him in 1994 and how by using the new metric, which will be described below, one may make a round trip between 2 celestial objects in an arbitrary small time as measured by an observer that remained at rest during the trip.

2.1 Metric of the Warp Drive

Before one can move on to the more technical stuff, the basic idea for the thought of this **Warp Drive** will be quoted. For this reason, an example of the inflationary phase of the early universe will be used and if the relative speed of the separation of two co-moving observers will be considered. It can be easily understood that if the relative speed of these two observers is defined as:

$$u^{\alpha} = \frac{dx^{\alpha}}{d\tau} \tag{2.1}$$

A value will be obtained that is much larger than the speed of light. That of course doesn't mean that the observers are travelling faster than light but it shows that this enormous speed that was found is coming from the expansion of space-time itself. So the **Warp Drive** will be based in this example , in the sense that anyone can use the expansion of space-time to move away from a specific object at an arbitrary large velocity and also to approach an object at a same arbitrary large velocity.

Now one is ready to expand this idea and use a metric of space-time in the 3+1 formalism of [GR] as have been learnt previously on **Chapter 1**. So the metric of space-time can be written as:

$$ds^2 = -d\tau^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j \tag{2.2}$$

Now the terms of the metric must be defined. A Cartesian coordinate system will be chosen and it will be assumed that the spaceship is moving along the x-axis. For the **Warp Drive** to work a metric must be constructed that will push the spaceship along a trajectory, which will be described by an arbitrary function of time $x_s(t)$. One of many metrics that can give the desired effect is the following:

$$\alpha = 1 \tag{2.3a}$$

$$\beta^x = -u_s(t)f(r_s(t)) \tag{2.3b}$$

$$\beta^y = \beta^z = 0 \tag{2.3c}$$

$$\gamma_{ij} = \delta_{ij} \tag{2.3d}$$

where we have also defined:

$$u_s(t) = \frac{dx_s(t)}{dt} \tag{2.4a}$$

$$r_s(t) = [(x - x_s(t))^2 + y^2 + z^2]^{1/2} \tag{2.4b}$$

Then if a specific metric must be found, which will satisfy the conditions that were set earlier a new arbitrary function $f(r_s)$ must be defined. In this first paper from Alcubierre this particular function has been chosen to take the following form:

$$f(r_s) = \frac{\tanh\left(\sigma(r_s + R)\right) - \tanh\left(\sigma(r_s - R)\right)}{2\tanh} \tag{2.5}$$

with $R, \sigma > 0$ two arbitrary parameters, which will be later correlated with the radius of the **Warp Bubble** and it's wall thickness.

Also it is helpful to show that the function $f(r)$ very rapidly approaches a 'top hat' function for large σ:

$$\lim_{\sigma\to\infty} f(r_s) = \begin{cases} 1 & \text{for } r_s \in [-R, R] \\ 0 & \text{if otherwise} \end{cases}$$

So with the above definitions the metric $reference$ can be rewritten:

$$ds^2 = -dt^2 + (dx - u_s f(r_s)dt)^2 + dy^2 + dz^2 \tag{2.6}$$

Seeing as the metric is written in this form the following observations can be made: Firstly from $\gamma_{ij} = \delta_{ij}$ it can be seen that the 3-geometry of the hypersurfaces is always flat. Secondly,the fact that the *lapse function* is given by $\alpha = 1$ implies that the time-like curves normal to these hypersurfaces are *geodesics*,i.e the *Eulerian Observers* are in **free fall**. Space-time however is **not** that due to the presence of a non-uniform shift. Closing with the observations, since the *shift vector* β^i vanishes for $r_s >> R$, it can be observed that at any time t, space-time will be flat everywhere, except within a region with a radius of order R centred at the point $(x_s(t), 0, 0)$.

2.2 Space-time of the Warp Drive

Now one can proceed further with the **Warp Drive** and discover some of it's very important properties. Since it has been shown that the 3-geometry is flat, the information about the *curvature* of space-time will be contained in the *extrinsic curvature* tensor K_{ij}. This tensor describes how the 3-dimensional hypersurfaces are embedded in 4-dimensional space-time and it is defined by:

$$K_{ij} = \frac{1}{2\alpha}(D_i\beta_j + D_j\beta_i - \frac{\vartheta g_{ij}}{\vartheta t}) \tag{2.7}$$

where D_i denotes the *covariant differentiation* with respect to the 3-metric γ_{ij}. Now if someone takes the forms ofα and γ_{ij}, it is not difficult to see that this expression reduces to:

$$K_{ij} = \frac{1}{2}(\vartheta_i\beta_j + \vartheta_j\beta_i) \tag{2.8}$$

Thus the expansion of the volume elements associated with the **Eulerian Observers** can be inserted, which is given as:

$$\theta = \alpha Tr(K) \tag{2.9}$$

and from this expression it is not difficult to show that:

$$\begin{aligned} \theta &= -u_s(\frac{\vartheta f}{\vartheta r_s}\frac{\vartheta r_s}{\vartheta y} + \frac{\vartheta f}{\vartheta r_s}\frac{\vartheta r_s}{\vartheta z}) \\ \rightarrow \theta &= u_s\frac{x_s}{r_s}\frac{df}{dr_s} \end{aligned} \tag{2.10}$$

Now for the next step, the relation between the *proper time* and the *coordinate time* of an observer is given by:

$$d\tau = \alpha dt \tag{2.11}$$

and seeing as $\alpha = 1$ has been chosen it can easily be discerned that $d\tau = dt$. That implies not only that the spaceship moves on a timelike curve, but also that it's *proper time* is equal to its *coordinate time*. Since *coordinate time* is also equal to the *proper time* of a stationary distant observer in the flat region of space-time a conclusion is being drawn that the spaceship suffers no dilation as it moves. However it is really worthwhile to note one more property of the spaceship's space-time. So it is really straightforward to prove that the spaceship moves on a geodesic, as has been talked about earlier. This means that even though the coordinate acceleration can be an arbitrary function of time, the proper time acceleration along the spaceship's path with **always** be **zero**.

2.3 Using the metric for space travel

Now for the last step it will be demonstrated how the metric that was created can make a round trip between two stars A and B, which are separated by distance R, in an arbitrary small time. Initially, a space ship will be considered as making a trip starting from star A using it's fuel engines so that it has a velocity $u << 1$. When it arrives at a distance r from star A which is given by the relation $r << R$ the metric that was discussed earlier is being created with some arbitrary way unknown to the observer so that the space ship will now have a *coordinate acceleration* α towards star B that changes from 0 to α. Thus the ship will continue its trip and when its half way to B the space-time is being modified in such a way that the *coordinate acceleration* changes from α to $-\alpha$. Then it can be easily proven that the space ship will arrive at a distance d from star B,where it's velocity becomes $u_s = 0$. Finally the spaceship can proceed to star B again using it's fuel engines with a velocity $u << 1$.

The above theoretical construction is a very good example of using the newly created metric for space travel as it can be shown below. If the changes in the *coordinate acceleration* are defined as very rapid the total *coordinate time* T can be calculated for a one-way trip:

$$T = 2\left\{\frac{d}{u} + \sqrt{\frac{D-2d}{\alpha}}\right\} \tag{2.12}$$

and it can be also safely assumed that since both stars remain in flat space-time and since it was set $\alpha = 1$ that their proper time is equal to their coordinate time. But the proper time of an observer measured on the space ship is given by:

$$\tau = 2\left\{\frac{d}{\gamma u} + \sqrt{\frac{D-2d}{a}}\right\} \tag{2.13}$$

with $\gamma = (1-u^2)^{-1/2}$ the known Lorentz factor. It can be easily understood that the flat spacetime's effect on the space ship is the time dilation and this is taking effect only in the initial and final stages of the trip. Finally if the condition $r >> R$ holds then we have:

$$\tau \approx T \approx 2\sqrt{\frac{D}{\alpha}} \tag{2.14}$$

Hence it can be deduced that it is easy to make the duration of the trip as small as it is needed by increasing the value of α. Furthermore because a round trip takes twice as much time to be completed then the *proper time* measured from the space ship and an observer on a distant star can take an arbitrary small value.

Closing our chapter, while a great metric has been constructed, which allows space travel, some serious drawbacks have erupted mainly from the quantum mechanic effects of the **Warp Drive** as this configuration will be called from now on.

Chapter 3

Arising Problems in Warp Drive Metric

Previously, the newly constructed metric was described and it was explained how someone can achieve 'superluminal' travel by applying the metric on a round trip between two stars. The thing that wasn't taken care of though, is what would happen if someone tried to measure the energy of the space ship observer or the matter required for this metric to be a technologically viable solution. In this upcoming chapter we will delve deeper in the metric's 3+1 formalism and consequentially deal with the problems that will arise.

3.1 Description of the 3 Energy Conditions

The first step we are going to take, is check if the metric obeys the 3 energy conditions and if it doesn't , find out where the problems lie and search for a way to bypass them. However for us to proceed we first need to acquaint ourselves with the 3 energy conditions.

Firstly an energy condition is one of the various alternative conditions, which can be applied to the matter content of general relativity, when it is either neither possible nor desirable to specify the content of matter. Then the energy conditions are specified , when the stress-energy tensor is given by:

$$T^{\mu\nu} = diag(\rho, p_1, p_2, p_3) \tag{3.1}$$

where ρ is the mass density and p_i are the three principal pressures. It can be noted now that , when $p_1 = p_2 = p_3$ one takes the stress-energy tensor of the perfect fluid.

We are thus ready to proceed to the definitions of the 3 energy conditions,which are going to be illustrated below.

3.1.1 Null Energy Condition [NEC]

The Null Energy Condition, or [NEC], is an inequality that must hold for any *null vector* k^μ and it is demonstrated below:

$$T_{\mu\nu}k^\mu k^\nu \geq 0 \tag{3.2}$$

and if this inequality is applied to the stress-energy tensor, the result is:

$$\rho + p_i \geq 0 \tag{3.3}$$

3.1.2 Weak Energy Condition [WEC]

In addition, the Weak Energy Condition, or [WEC], concerns only *timelike vectors* of the form U^μ,which must satisfy the inequality :

$$T_{\mu\nu}U^\mu U^\nu \geq 0 \tag{3.4}$$

One may come to the conclusion that the [WEC] can be used for any timelike observer with 4-velocity U^μ for the calculation of his energy density. This particular energy condition will hold great value, because as it has been mentioned earlier, a timelike observer with 4-velocity of the form $n^\alpha = -\alpha(1,0,0,0)$ or $n_\alpha = \frac{1}{\alpha}(1,-\beta^i)$, is the **Eulerian Observer**. For a last comment, by continuity one can observe that the [WEC] implies the [NEC].

3.1.3 Strong Energy Condition [SEC]

Last but not least, the Strong Energy Condition, or [SEC], states that for any *timelike vector* U^μ, the inequality that must hold is:

$$\left(T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}\right) U^\mu U^\nu \geq 0 \tag{3.5}$$

Overall the [SEC] implies the [NEC] as it is a continuation of the [WEC] but that doesnt mean that is necessarily implies the [WEC]

3.1.4 Averaged Energy Conditions

However, for the energy conditions that were obtained earlier, violations have been observed mostly from quantum effects, such as the Casimir effect. The proposed solution was the averaging of these energy conditions over *time like* or *null* geodesics. The newly constraints are that the averaged conditions must simply hold over the integration along these geodesics. That being the case, the averaged energy conditions are somewhat weaker than the energy conditions.

3.1.4.1 Averaged Null Energy Condition [ANEC]

For the first step the [ANEC] is satisfied, when it is integrated along a *null* curve Γ and one can safely assume that:

$$\int_\Gamma T^{\mu\nu} k^\mu k^\nu d\lambda \geq 0 \tag{3.6}$$

where as was stated before k is the null vector and λ is the affine parameter.

3.1.4.2 Averaged Weak Energy Condition [AWEC]

Equally the [AWEC] is being satisfied, providing that the following inequality holds:

$$\int_\Gamma T_{\mu\nu} U^\mu U^\nu ds \geq 0 \tag{3.7}$$

where Γ is a timelike curve, s is a parametrization and U^μ is the tangent vector onto Γ.

3.1.5 Volume Integral Quantifier

Unfortunately, as it has been shown, the integrals of the [ANEC] and the [AWEC], are being calculated over a curve, so a volume integral cant be computed, which gives information for the energy distribution needed to violate the energy conditions. This led Visser to propose a so called 'volume-integral quantifier', which in term gives information about the negative energy densities in the parts where the following integrals become negative:

$$\int T_{\mu\nu}U^{\mu}U^{\nu}dV \text{and} \int T_{\mu\nu}k^{\mu}k^{\nu}dV \tag{3.8}$$

3.2 Energy conditions Applied to the Warp Drive metric

Subsequently, the first energy condition that will be put to the test is the [WEC], as it concerns timelike vectors, the reason for this being that one can easily discern from the **Warp Drive** metric that an observer on the space ship can be given in the form of an **Eulerian Observer** with U described by either $U^{\mu} = -\alpha(1,0,0,0)$ or $U_{\mu} = \frac{1}{\alpha}(1,\beta^{i})$.

3.2.1 Violation of the [WEC]

Having considered the observer's properties Equation(3.4) is ready to be calculated, where $T_{\mu\nu}$ will be obtained from the metric in Equation(2.6) and the timelike vector U^{μ} will be taken from the **Eulerian Observers**. As a result one gets:

$$T_{\mu\nu}U^{\mu}U^{\nu} = -\alpha(1,0,0,0) - \alpha(1,0,0,0)T^{00} = a^{2}T^{00} \tag{3.9}$$

However is is already shown from tensor analysis in [GR] ,that the stress-energy tensor $T^{\alpha\beta}$ is correlated to the Einstein tensor $G^{\alpha\beta}$ with the following relation:

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta} \tag{3.10}$$

Eventually if the correct replacements are made in Equation(3.9), the Einstein tensor $G^{\alpha\beta}$ is calculated for $\alpha, \beta = 0$ we will obtain for the [WEC]:

$$T_{\mu\nu}U^{\mu}U^{\nu} = \alpha^2 T^{00} = \frac{1}{8\pi}G^{00} = -\frac{u^2}{32\pi}\left(\left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) \tag{3.11}$$

However one can replace f function with the given one from Equation(**??**) and using that with some more operations the next equation is concluded :

$$\begin{aligned} T_{\mu\nu}U^{\mu}U^{\nu} &= -\frac{u^2}{32\pi}\left(\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\right)^2\right) \\ &= -\frac{u^2}{32\pi}\left(\left(\frac{\partial f}{\partial r}\frac{y}{r}\right)^2 + \left(\frac{\partial f}{\partial r}\frac{z}{r}\right)^2\right) \\ &= -\frac{1}{32\pi}\frac{u^2(y^2+z^2)}{r^2}\left(\frac{\partial f}{\partial r}\right)^2 \leq 0 \end{aligned} \tag{3.12}$$

According to this equation for the [WEC] in the **Warp Drive** metric some useful conclusions can be stated.

I For one thing, it can be easily observed that the application of the [WEC], gives a measured energy density during the space travel that its nowhere positive

II Moreover one can calculate the energy density in a toroidal region around the x-axis, which is also the direction of the travel and it is shown in the following figure:

III As a final point if a generic *lapse function* $a = (t, x, y, z)$ is inserted into Equation(3.11), it then becomes:

$$T_{\mu\nu}U^{\mu}U^{\nu} = -\frac{u^2}{32\pi\alpha}\left(\left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) \tag{3.13}$$

To conlude with the [WEC] violations, it is easily understood from the last equation that if one sets a to be equal to both sides of the **Warp Drive** metric, so that *proper time* equals *coordinate time* , the negative energy density required can be decreased if extremely large values for a are imposed. The problem that arises then is that for extremely large values of the *lapse function* a, following the

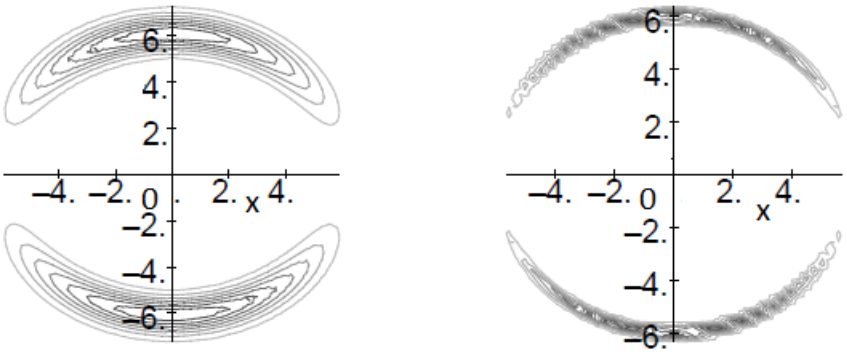

FIGURE 3.1: The figure shows the toroidal distribution of the energy density, perpendicular to the direction of the trip which is directed towards the x-axis. For the two figures the values chosen are, for u=2 and R=6 ,with $\sigma = 1$ and $\sigma = 4$,respectively

Equation $d\tau = adt$ the *proper time* measured inside the **Warp Bubble** walls will be absurdly large.

3.2.2 Total Amount of Energy

As was stated in Chapter (3.1.6), if someone wants to find the total amount of matter in the **Warp Drive** that is needed to violate the energy conditions one must apply the volume integral quantifier from Equation(3.8) to the newly obtained results.

$$\begin{aligned} M_{warp} = \int \rho_{warp} d^3x &= \int T_{\mu\nu} U^\mu U^\nu d^3x \\ &= -\frac{u^2}{32\pi} \int \frac{y^2+z^2}{r^2} \left(\frac{\partial f}{\partial r}\right)^2 r^2 dr d^2\Omega \\ &= -\frac{u^2}{12} \int \left(\frac{\partial f}{\partial r}\right)^2 r^2 dr \end{aligned} \tag{3.14}$$

It must be stated that this particular integral doesn't give the total mass of the space-time but the amount of negative energy needed to be localized in the **Warp Bubble** walls. To conclude with the mass, if the the form function f is replaced with the form function from Equation(2.5), the following relation is given for the M_{warp}:

$$M_{warp} \approx u^2 R^2 \sigma \tag{3.15}$$

Under those circumstances, it is understood that the total matter of negative energy needed for the *Warp Drive* metric to work, scales quadratically with the velocity of the space ship u and the radius R of the **Warp Drive** bubble and inversely with the thickness of the **Warp Bubble** walls.

3.2.3 Violation of the [NEC]

To conclude with the energy violations from the **Warp Drive** metric, the [NEC] must be inspected. As was shown before the [NEC] states that $T_{\mu\nu}k^{\mu}k^{\nu} \geq 0$, where $T_{\mu\nu}$ is the stress-energy tensor and k^{μ} is a null vector. Consequently the [NEC] becomes for a null vector on the direction:

$$T_{\mu\nu}k^{\mu}k^{\nu} = -\frac{u^2}{8\pi}\left(\left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) \pm \frac{u}{8\pi}\left(\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) \tag{3.16}$$

and if this particular result is averaged on the $\pm x$ direction then it is obtained:

$$\frac{1}{2}\{T_{\mu\nu}k^{\mu}_{+x}k^{\nu}_{+x} + T_{\mu\nu}k^{\mu}_{-x}k^{\nu}_{-x}\} = -\frac{u^2}{8\pi}\left(\left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) \tag{3.17}$$

In that case it is understood that [NEC] is violated for all u. Furthermore, even if the conditions were not averaged, the coefficient term linear in u must be nonzero somewhere in space time. More importantly if the Alcubierre form function f is taken from Equation(2.5) and then replaced inside Equation(3.16), the following result is produced:

$$T_{\mu\nu}k^{\mu}k^{\nu} = -\frac{1}{8\pi}\frac{u^2(y^2+z^2)}{r^2}\left(\frac{\partial f}{\partial r}\right)^2 \pm \frac{u}{8\pi}\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial}{\partial z}\frac{\partial f}{\partial z}\right) \tag{3.18}$$

The calculation for the $2nd$ part of the equation is given by:

$$\begin{aligned}
&\pm\frac{u}{8\pi}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial y}\right) + \left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\right)\right)\right.\\
&\Rightarrow \pm\frac{u}{8\pi}\left(\left(\frac{\partial}{\partial y}\frac{\partial f}{\partial r}\right)\frac{\partial r}{\partial y} + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial y^2} + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial r}\right)\frac{\partial r}{\partial z} + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial z^2}\right)\\
&\Rightarrow \pm\frac{u}{8\pi}\left(\frac{d^2 f}{dr^2}\left(\left(\frac{\partial r}{\partial y}\right)^2 + \left(\frac{\partial r}{\partial z}\right)^2\right) + \frac{df}{dr}\left(\frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2}\right)\right)
\end{aligned} \tag{3.19}$$

Finally Equation(3.16) takes the final form:

$$T_{\mu\nu}k^{\mu}k^{\nu} = -\frac{u^2}{8\pi}\frac{u^2(y^2+z^2)}{r^2}\left(\frac{df}{dr}\right)^2 \pm \frac{u}{8\pi}\left(\frac{(x-x_o(t))^2+y^2+z^2}{r^2}\frac{df}{dr} + \frac{y^2+z^2}{r^2}\frac{d^2f}{dr^2}\right) \tag{3.20}$$

From this final form of the Equation for the violations of the [NEC], some conclusions can be drawn. As it is easily understood the first term of Equation(3.20) is everywhere negative on the **Warp Drive** space-time. The first derivative of f ,$\frac{df}{dr}$, is also everywhere negative for the simple reason that the form function is set to be 1 at the centre of the **Warp Bubble** and 0 at the outside. Lastly the second derivative of f ,$\frac{d^2f}{dr^2}$, is again everywhere negative because the form function obtains its maximum value at the centre of the **Warp Bubble**.

3.3 Quantum Inequality Application to Warp Drive Metric

Finally the energy violations on the **Warp Drive** metric have been discussed and they have been fully understood. There is also, however a further point to be considered in the form of the **Quantum Inequality**, or [QI], as it is going to be referred to from now on. For that reason the next steps are going to concern the understanding of the [QI] and its application to the metric.

3.3.1 Quantum Inequality

Work by Ford and Pfennig in the 70's led to a discovery of a new set of energy constraints. After some considerable work on this new set of constraints, they in turn led to negative energy fluxes. Further details will not concern us on the way, which the [QI] was produced, simply because it involves some very complex mathematics. It will only be stated that the [QI] is proven directly from QFT and in 4-dimensional Minkowski spacetime, for free quantized massless scalar fields, takes the form:

$$\frac{t_o}{\pi}\int_{-\infty}^{+\infty}\frac{< T_{\mu\nu}U^{\mu}U^{\nu} >}{t^2+t_o^2}dt \geq -\frac{3}{32\pi^2 t_o^4} \tag{3.21}$$

where t_o is the time scale of the sampling.

3.3.2 Quantum Inequality Application

As it is easily shown from the last Equation, a familiar term is recognized in the form of the Equation(3.11). That being the case one can insert the **Warp Drive** metric into the [QI] and observe the results concerning negative energies. Thus if Equation(3.11) is inserted into Equation(3.21) the following result is produced:

$$t_o \int_{-\infty}^{+\infty} \frac{u(t)^2}{r^2} \left(\frac{df}{dr}\right)^2 \frac{dt}{t^2 + t_o^2} \leq \frac{3}{\rho^2 t_o^4} \tag{3.22}$$

where $\rho = (y^2 + z^2)^{\left(\frac{1}{2}\right)}$ was set for convenience. With the last equation in mind, one can now follow the important work done by Ford and Pfennig and obtain some useful properties concerning the **Warp Drive** metric in the quantum mechanics regime. To begin with one must take the following steps:

• Initially the velocity $u(t)$ may be considered to be constant $u(t)^2 = u_b$.

• Following the first step, one can set the time scale of the sampling to be considered small compared to the time scale of the change in the **Warp Bubble's** velocity. Then for this small time scale the term $(t^2 + t_o^2)^{-1}$ becomes strongly peaked, so unfortunately one can only obtain results for a small part of the space ship's geodesic.

• Finally, if t is set to be 0, then the calculations are made on the equator of the geodesic so one can set:

$$x \approx f(\rho) u_b t \tag{3.23}$$

and then :

$$r(t) = \left((u_b t)^2 \left(f(\rho) - 1\right)^2 + \rho^2\right)^{\frac{1}{2}} \tag{3.24}$$

Now for the next logical step, one may consider a slightly different form function than the one given by Alcubierre in Equation(2.5).

The new form function will be considered as a piece-wise continuous form function that is given by:

$$f_{p\cdot c\cdot}(r) = \begin{cases} 1 & \text{for } r < R - \dfrac{\Delta}{2} \\ -\dfrac{1}{\Delta}\left(r - R - \dfrac{\Delta}{2}\right) & \text{for } R - \dfrac{\Delta}{2} < r < R + \dfrac{\Delta}{2} \\ 0 & \text{for } r > R + \dfrac{\Delta}{2} \end{cases}$$

where R is the radius of the **Warp Bubble** and Δ is the thickness of the **Warp Bubble** wall. A relation can be given for Δ if one takes the aforementioned piece-wise form function to be equal to the form function from Equation(2.5) at $r = R$. Then:

$$\Delta = \frac{\left(1 + \tanh^2(\sigma R)\right)^2}{2\sigma \tanh(\sigma R)} \quad (3.26)$$

To return to the subject of the [QI], one can observe that in the limit of big σR the approximation $\Delta = \dfrac{2}{\sigma}$ can be taken, so that the Equation(3.22) is reduced to:

$$t_o \int_{-\infty}^{+\infty} \frac{dt}{\left(t^2 + \bar{\beta}^2\right)\left(t^2 + t_o^2\right)} \leq \frac{3\Delta^2}{u_b^2 t_o^4 \bar{\beta}^2} \quad (3.27)$$

where once again for convenience $\bar{\beta}$ is set as $\bar{\beta} = \dfrac{\rho}{u_b\left(1 - f(\rho)\right)}$. If the integral part of the inequality is evaluated, a new form of the [QI] is given by:

$$\frac{\pi}{3} \leq \frac{\Delta^2}{u_b^2 t_0^4}\left(\frac{u_b t_o}{\rho}\left(1 - f(\rho)\right) + 1\right) \quad (3.28)$$

Following the analysis done by Ford and Pfennig, having considered the final form of the application of the [QI], as shown before, one may now calculate the *Riemann tensors* so that a final equation for the **Warp Bubble's** wall thickness can be obtained. It was proven that the largest component of the *Riemann tensors* is :

$$\mid R^{\wedge\wedge\wedge\wedge}_{t\ y\ t\ y} \mid = \frac{3u_b^2 y^2}{4\rho^2}\left(\frac{df(\rho)}{d\rho}\right)^2 \quad (3.29)$$

Consequently, if one knows the largest *Riemann tensors* component, one may also know the minimum radius of the **Warp Bubble**, given by:

$$r_{min} = \frac{1}{\mid R^{\wedge\wedge\wedge\wedge}_{t\ y\ t\ y} \mid} \approx \frac{2\Delta}{\sqrt{3}u_b} \tag{3.30}$$

Accordingly, if some considerations are made such as, $y = \rho$, t_o must be smaller from the radius by a factor α and if the new piece-wise form function is used, one can finally arrive at the conclusion for the time scale of the sampling time:

$$t_0 = \alpha \frac{2\Delta}{\sqrt{3}u_b} \tag{3.31}$$

For the final step if the sampling time in now inserted into Equation(3.28) and if one considers that $\frac{\Delta}{\rho} \approx \frac{u_b t_o}{\rho} \ll 1$ and ignores the term $1 - f(\rho)$ the useful inequality for the **Warp Bubble's** wall thickness is derived:

$$\begin{aligned} \Delta &\leq \frac{3}{4}\sqrt{\frac{3}{\pi}}\frac{u_b}{\alpha^2} \\ \Delta &\leq 10^2 u_b L_{planck} \end{aligned} \tag{3.32}$$

So as the final inequality states, the wall thickness of the **Warp Bubble** must be in the order of the *Planck Length*, unless the **Warp Bubble's** velocity is extremely large.

3.3.3 Total Negative Energy Requirements

There is also however a further point to be considered and that is the total negative energy required to sustain the **Warp Drive** metric. In section (3.1.6) a volume integral quantifier was defined that can help calculate the total negative energy needed. Therefore, if for simplicity a constant velocity was set, so that $x_x(t) = u_b$, and if the total energy is conserved and is everywhere constant we are going to set for $t = 0$:

$$r_s(t = 0) = \left(x^2 + y^2 + z^2\right)^2 \tag{3.33}$$

In this manner if Equation(3.22)and Equation(3.11) are inserted into Equation(3.8) the following result for the energy E is given:

$$E = \int \sqrt{\mid g \mid} < T^{00} > d^3x = -\frac{u_b^2}{32\pi} \int \frac{\rho^2}{r^2} \left(\frac{df(r)}{dr} \right)^2 d^3x \tag{3.34}$$

where $g = Det \mid g_{ij} \mid$ is the determinant of the spatial metric on the hypersurfaces Σ_t with constant t. Moreover, if the Cartesian coordinates are transformed to spherical the following result is produced:

$$E = -\frac{1}{12} u_b^2 \int_0^{+\infty} r^2 \left(\frac{df(r)}{dr} \right)^2 dr \tag{3.35}$$

For the next step if the newly piece-wise form function $f_{p.c.}$ is used it is easy to derive that the only contribution is being given from the bubble wall region, so that:

$$\begin{aligned} E &= -\frac{1}{12} u_b^2 \int_{R-(\Delta/2)}^{R+(\Delta/2)} r^2 \left(-\frac{1}{\Delta} \right)^2 dr \\ &= -\frac{1}{12} u_b^2 \left(\frac{R^2}{\Delta} + \frac{\Delta}{12} \right) \end{aligned} \tag{3.36}$$

It can be seen from the above analysis that for a **Warp Drive** metric to be technologically viable we need to set the radius R at least 100 meters so a medium ship can be fit into. Furthermore as was proven in the previous section there exists a certain limitation for the **Warp Bubble's** wall thickness in the form of Equation(3.32). That being said, one can neglect the second term of the integral and conclude for the negative energy:

$$\begin{aligned} E &\leq -6.2 * 10^{70} u_b L_{planck} \\ E &\leq -3 * 10^{20} M_{galaxy} u_b \end{aligned} \tag{3.37}$$

This result as one can understand is very interesting and very troublesome at the same time. Not only do we need negative energy of the orders of mass of entire galaxies but we also need it to be negative.

Closing this particular section it is important however not to assume the applicability of the restrictions provided by the[QI] in all cases. The reasons for this are, first that these restrictions are given, if one relies on the *Quantum Field Theory* to provide the exotic matter needed for the **Warp Drive** to work.However there are classical systems that while they violate the [WEC] and the [NEC] they dont provide violations or they fare up great against the [QI]. Second, even if the [QI] application on the **Warp Drive** metric place some very powerful restraints on our space time it nevertheless doesn't rule up the existence of such a space time. Finally for a closing statement one may hold the last inequality close at hand, because it will be the basis of the analysis in Chapter 5 and of our try to minimise the energy requirements of the **Warp Drive**.

3.4 Linearized Gravity Applied to the Warp Drive

Following the previous analysis on the **Warp Drive** metric in the quantum field regime, some serious drawbacks were observed and expanded. Also some powerful constrictions were set to the **Warp Bubble's** wall thickness and to the velocity of a hypothetical space ship. The thing is so far the space ship was treated as a particle with no mass and by consequence no contribution to the gravitational field. In the following section the **Warp Drive** metric will face some new 'enemies' in the form of the linearized gravity theory and in the form of a space ship with mass and the properties that come with it.

Consequently after the last statements the **Warp Drive** metric must initially be put through the weak field regime, where a finite mass space ship exists, which moves with a velocity on the x-axis,of the form $u \ll 1$. Before the linearized theory is ready to be applied though, one must first remember the form of the metric given by Equation(2.6) and carefully study the properties of the linearized gravity in Appendix A. Then for a last step it must be stated, that if the space ship is more than a test particle, it was forced earlier that $f = 0$ outside of the **Warp Bubble** and this implies that the geometry is *asymptotically Minkowski*.

3.4.1 [WEC] in Linearized Gravity

The first analysis that will be completed for the linearized gravity will observe how the newly constructed metric will fare up against the [WEC], as it was set earlier on this chapter. In linearized gravity the space time metric is given by:

$$ds^2 = (\eta_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{3.38}$$

where $h_{\mu\nu}$ and $\eta_{\mu\nu}$ are given respectively, by $h_{\nu \ll 1}$ and $\eta_{\mu\nu} = diag(-1, 0, 0, 0)$. As a result and having Equation(2.6) in mind $(h_{\mu\nu})$ takes the following form:

$$(h_{\mu\nu}) = \begin{bmatrix} u^2 f^2 & 0 & 0 & -uf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -uf & 0 & 0 & 0 \end{bmatrix} \tag{3.39}$$

The application of the linearized theory gives accurate results only for a first order approximation of u. That means that the term with u^2 inside $(h_{\mu\nu})$ must be ignored. For this reason the previous matrix is denoted to:

$$(h_{\mu\nu}) = \begin{bmatrix} 0 & 0 & 0 & -uf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -uf & 0 & 0 & 0 \end{bmatrix} \tag{3.40}$$

It is easily shown that the trace of $(h_{\mu\nu})$ is identically null, $h = h^\mu_\mu = 0$. Therefore the trace reverse of $(h_{\mu\nu})$ as is given in Appendix A takes the form, $\bar{h}_{\mu\nu} = h_{\mu\nu}$. Additionally, the following equations will be helpful when the [WEC] is going to be calculated in the frame of linearized gravity:

$$\frac{\partial^2 \bar{h}_{01}}{\partial x^\alpha \partial x^\beta} = -u \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} \tag{3.41}$$

and if the form function given from Equation(2.5) is inserted into the above equation:

$$\frac{\partial^2 \bar{h}_{01}}{\partial x^\alpha \partial x^\beta} = -u \left(\frac{\partial r}{\partial x^\alpha} \frac{\partial r}{\partial x^\beta} \frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 r}{\partial x^\alpha \partial x^\beta} \frac{df}{dr} \right) \tag{3.42}$$

It is of great interest now to use the *linearized Eintein Tensor*, obtained in Appendix A and insert it into the [WEC] given also by Equation(3.4). In linearized

gravity the velocity of an observer is the well known velocity of an **Eulerian Observer** in the form of $U = (-1, 0, 0, 0)$. Therefore as it has been shown numerous times the [WEC] reduces to:

$$T_{\mu\nu}U^{\mu}U^{\nu} = T_{00} = \frac{1}{8\pi}G_{00} \tag{3.43}$$

However the *linearized Einstein Tensor* is given by Equation(A.7)if we set $\mu, \nu = 0$:

$$G_{00} = -\frac{1}{2}\left(\bar{h}_{00,\mu}{}^{\mu} - \bar{h}_{\mu\nu,}{}^{\mu\nu} - 2\bar{h}_{0\mu,0}{}^{\mu}\right) \tag{3.44}$$

The respective terms are given by:

$$\bar{h}_{00,\mu}{}^{\mu} = 0 \tag{3.45a}$$

$$\bar{h}_{\mu\nu,}{}^{\mu\nu} = 2\bar{h}_{01,01} \tag{3.45b}$$

$$\bar{h}_{0\mu,0}{}^{\mu} = \bar{h}_{01,01} \tag{3.45c}$$

It is important to note however that, in general, if second time derivatives appear, they can be neglected because $\frac{\partial}{\partial t}$ is of the same order as $u\frac{\partial}{\partial x}$, so that the relation $\partial_{\mu}\partial^{\mu} = \nabla^2 + O(u^2\nabla^2)$ holds. Also it is possible to prove that both G_{00} and G_{0i} don't contain second time derivatives of any generic $\bar{h}_{\mu\nu}$ and that was done by Schultz. As a result six equations $G_{ij} = 8\pi T_{ij}$ are true dynamical equations and in contrast the equations $G_{0\mu} = 8\pi T_{0\mu}$ are not dynamical and are called *constraint equations*. The reason for that being is that they correlate the initial data for the other six equations with the equations themselves so it is impossible to choose arbitrary initial data. In this manner if one computes the subequations using Equation(3.42) and then substitutes the results into the *linearized Einstein Tensor* one obtains:

$$G_{00} = O(u^2) \tag{3.46}$$

and finally Equation(3.4) becomes:

$$T_{\mu\nu}U^{\mu}U^{\nu} = T_{00} = O(u^2) \tag{3.47}$$

Thus it was shown for a first analysis that the [WEC] is identically saturated and no problems arise.

3.4.2 Space Ship inside the Warp Bubble

In this sections final analysis, one must consider a space ship in the interior of the **Warp Bubble**. If the space ship is moving with a non-relativistic velocity $u \ll 1$ on the x-axis then the metric is given by:

$$\begin{aligned} ds^2 =& -dt^2 + dy^2 + dz^2 + (dx - uf(x-ut,y,z)dt)^2 \\ & - 2\Phi(x-ut,y,z)\left(dt^2 + dy^2 + dz^2 + (dx - uf(x-ut,y,z)dt)^2\right) \end{aligned} \tag{3.48}$$

where Φ is the gravitational field of the space ship. Here it must be noted that when $\Phi = 0$ then the metric is the same as the metric in Equation(2.6) and when $u = 0$ the space time is the same one produced by a stationary source. Furthermore, if the transformation $\bar{x} = x - ut$ is applied to the previous metric one obtains:

$$\begin{aligned} ds^2 =& -dt^2 + dy^2 + dz^2 + (d\bar{x} - u\left(f(\bar{x},y,z) - 1\right)dt)^2 \\ & - 2\Phi(\bar{x},y,z)\left(dt^2 + dy^2 + dz^2 + (d\bar{x} - u\left(f(\bar{x},y,z) - 1\right)dt^2\right) \end{aligned} \tag{3.49}$$

The following instances can be taken if the form function f takes the values $f \to 1$ for an observer inside the **Warp Bubble** and if $f \to 0$ for an observer outside the **Warp Bubble**.

$$ds^2 = -dt^2 + dy^2 + dz^2 + d\bar{x}^2 - 2\Phi(\bar{x},y,z)\left(dt^2 + d\bar{x}^2 + dy^2 + dz^2\right) \tag{3.50}$$

and

$$\begin{aligned} ds^2 =& -dt^2 + dy^2 + dz^2 + (d\bar{x} + udt)^2 \\ & - 2\Phi(\bar{x},y,z\left(dt^2 + dy^2 + dz^2 + (d\bar{x} + udt)^2\right) \end{aligned} \tag{3.51}$$

As the first part of the analysis of the space time containing a finite mass space ship is concluded, one is now ready to proceed to the first and second order approximations of this particular space time.

3.4.2.1 First Order Approximation

Now if linearised theory is applied, which means the terms linear to u and Φ will be kept intact and all terms of superior order will be neglected, the element $(h_{\mu\nu})$ of the metric (3.48) is given:

$$(h_{\mu\nu}) = \begin{bmatrix} -2\Phi & 0 & 0 & -uf \\ 0 & -2\Phi & 0 & 0 \\ 0 & 0 & -2\Phi & 0 \\ -uf & 0 & 0 & -2\Phi \end{bmatrix} \tag{3.52}$$

where the trace of the matrix is $h = h^{\mu}_{\mu} = -4\Phi$ and the reverse matrix needed for further computations $\bar{h}_{\mu\nu}$ takes the form:

$$(\bar{h}_{\mu\nu}) = \begin{bmatrix} -4\Phi & 0 & 0 & -uf \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -uf & 0 & 0 & 0 \end{bmatrix} \tag{3.53}$$

Following the previous results and from the [WEC], $T_{\mu\nu}U^{\mu}U^{\nu}$, the trace reversed components of the *linearized Einstein Tensor* that need to be computed from Equation(3.44) are:

$$\bar{h}_{00,\mu}{}^{\mu} = -\bar{h}_{00,00} + \bar{h}_{00,11} + \bar{h}_{00,22} + \bar{h}_{00,33} \tag{3.54a}$$

$$\bar{h}_{\mu\nu,}{}^{\mu\nu} = \bar{h}_{00,00} - 2\bar{h}_{01,01} \tag{3.54b}$$

$$\bar{h}_{0\mu,0}{}^{\mu} = -\bar{h}_{00,00} + \bar{h}_{01,01} \tag{3.54c}$$

Then if these particular subequations are replaced into Equation(3.44):

$$G_{00} = 2\nabla^2\Phi + 0(u^2, u\Phi, \Phi^2) \tag{3.55}$$

And taking *Poisson's Equation*,$\nabla^2\Phi = 4\pi\rho$, we conlcude for the [WEC],$T_{\mu\nu}U^{\mu}U^{\nu}$:

$$T_{\mu\nu}U^{\mu}U^{\nu} = \rho + 0(u^2, u\Phi, \Phi^2) \tag{3.56}$$

where ρ is the energy density of the space ship, that was inserted into the **Warp Bubble** which is also identically positive. However it is a good time to state that in linearized theory the total ADM mass of the space time simply reduces to that

of the space ship. For that reason and after using the 'Volume Integral Quantifier' from Equation(3.8), one can easily conclude for the mass of the space ship:

$$M_{\mathrm{ADM}} = \int T_{00} d^3x = \int \rho d^3x + \int O(u^2, u\Phi, \Phi^2) d^3x = M_{ship} + O(u^2, u\Phi, \Phi^2) \tag{3.57}$$

As a result it is easily shown that for a first order approximation the dominant term is positive and the [WEC] is once again satisfied.

3.4.2.2 Second Order Approximation

Subsequently, one can keep the exact u dependence as before but linearize the gravitational field of the space ship Φ leading to a second order approximation. Inserting the *linearized Einstein Tensor*:

$$G_{00} = -\frac{u^2}{4}\left(\left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) + 2\nabla^2\Phi + O(\Phi^2) \tag{3.58}$$

and constituting the *Poisson's Equation* and then inserting the result into [WEC] from Equation(3.4):

$$T_{\mu\nu}U^\mu U^\nu = \rho - \frac{u^2}{32\pi}\left(\left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right) + O(\Phi^2) \tag{3.59}$$

For the next step if the Alcubierre form function is considered from Equation(2.5) as it has been shown numerous times:

$$T_{\mu\mu}U^\mu U^\nu = \rho - \frac{1}{32\pi}\frac{u^2(y^2+z^2)}{r^2}\left(\frac{df}{dr}\right)^2 + O(\Phi^2) \tag{3.60}$$

Finally once again the 'Volume Integral Quantifier' can be used and this will result in:

$$\begin{aligned} \int T_{\mu\mu}U^\mu U^\nu d^3x &= M_{\mathrm{ship}} - u^2R^2\sigma + \int O(\Phi^2) d^3x \\ M_{\mathrm{total}} &= M_{\mathrm{ship}} + M_{\mathrm{warp}} + \int O(\Phi^2) d^3x \end{aligned} \tag{3.61}$$

Closing this analysis of the [WEC] if one logically sets the 'Volume Integral Quantifier' to be positive, then by default:

$$u^2 R^2 \sigma \lesssim M_{ship} \tag{3.62}$$

where this inequality denotes the logical assumption that the total energy density stored in the **Warp Drive** space time must be less than the total mass-energy stored in the space ship. For a clearer assumption of the restraints that are being put on the velocity of the **Warp Drive** from the [WEC] one can rewrite the inequality in terms of the radius of the space ship,R_{ship} and the thickness of the **Warp Bubble's** wall, $\Delta = \frac{1}{\sigma}$:

$$u^2 \lesssim \frac{M_{ship}}{R_{ship}} \frac{R_{ship}\Delta}{R^2} \tag{3.63}$$

Evidently this inequality states that if someone wants to have a large **Warp Drive** velocity one must always follow these constrictions put from the previous analysis on the space ship's mass and radius.

Chapter 4

Limiting the Energy Violations

On the previous chapters a new metric was constructed, which allowed a certain definition of super-luminal travel to occur when the hypothesised spaceship moved on the time-like geodesics of this metric. Afterwards, this new metric was put to the tests of the quantum mechanics and the weak-field regimes of physics, where the problems of our **Warp Drive** metric were made clear; in the form of excessive quantities of negative energy required for the metric to be technologically achievable and in the form of constraints set to the velocity and the mass of the spaceship and also limits set to the **Warp Bubble's** wall thickness and radius. In this last chapter we will follow the analysis done by Van den Broeck in 1999, where by introducing a new term in our metric we will achieve to limit the negative energy required from the order of the mass of our entire universe to the order of some solar masses.

4.1 Van Den Broeck's Proposed Metric

After studying the Alcubierre metric, as shown in Equation(2.6), van den Broeck came to the conclusion that a trivial modification to the existing metric can have dramatic consequences and results on the total negative energy required to operate the **Warp Drive**, given by the application of the [WEC] on the previous chapter. The 'trick' that was proposed to limit the total negative energy was to keep the *surface area* of the **Warp Bubble** microscopically small while at the same time expand the *spatial volume* inside the **Warp Bubble**. Eventually it was found that the most natural way to achieve this, was to enter a new term in Equation(2.6) thus taking the form:

$$ds^2 = -dt^2 + B(r_s)^2 \left((dx - u_s(t)f(r_s)dt)^2 + dy^2 + dz^2\right) \tag{4.1}$$

where for convenience the velocity will be taken constant,$u_s(t) = u_b$.
For this method to work and have the desirable effect on the metric it was set that $B(r_s)$ would be a twice differentiable function such that, for some $\bar{R}$ and $\bar{\Delta}$:

$$B(r_s) = \begin{cases} 1+\alpha & \text{for } r_s < \bar{R} \\ \in (1, 1+\alpha] & \text{for } \bar{R} \leq r_s < \bar{R} + \bar{\Delta} \\ 1 & \text{for } \bar{R} + \bar{\Delta} \leq r \end{cases}$$

Also for the form function f it was chosen:

$$f(r) = \begin{cases} 1 & \text{for } r_s < R \\ \in (0, 1] & \text{for } R \leq r_s < R + \Delta \\ 0 & \text{for } R + \Delta \leq r_s \end{cases}$$

The next figure shows the areas where B and f vary and give some important information about these areas.

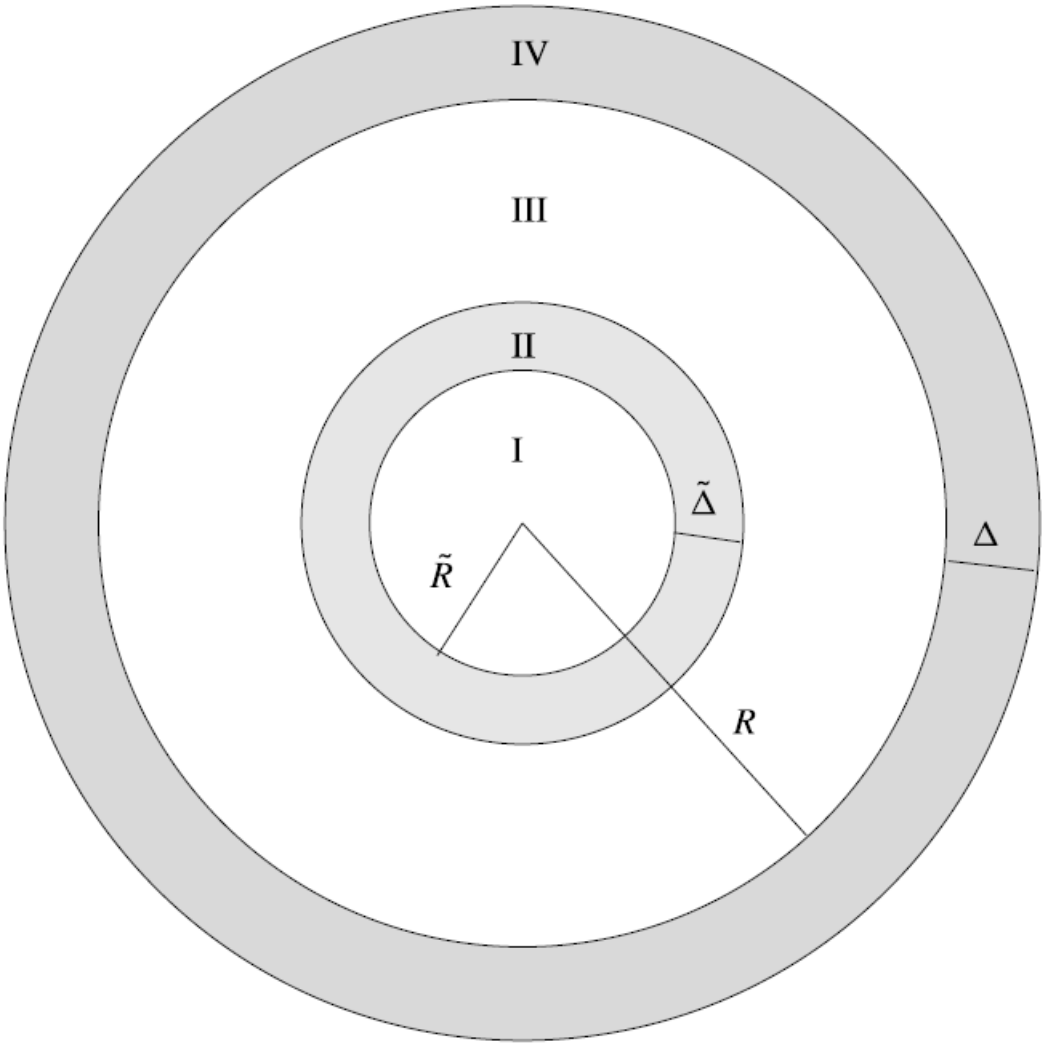

FIGURE 4.1: In this figure one can discern four different regions. In region I is where one has a large innre metric diameter.In region II is where the transition is happening from the blown up part of space to the normal space. It is also where B varies. From region III and onward one can find the original Alcubierre metric. Finally in region IV one comes across the wall of the **Warp Bubble**. It is where f varies. It must be noted that spacetime is everywhere flat, except the shaded regions

4.2 Calculating the [WEC] with the new Metric

Having studied the new metric and therefore the newly constructed spacetime and after understanding its new properties one is ready to try and calculate the [WEC]. From the previous analysis in chapter 4 it was shown that the **Warp Bubble's** wall thickness is constrained by the Inequality(3.32). To help the reader remember the inequality, it was the one that constrained the **Warp Bubble's** wall thickness to the order of the *Planck Length*.

Additionally it was shown that where the original Alcubierre metric takes hold, meaning the region III-IV and also where B=1 in the newly constructed metric, the energy was calculated:

$$E_{IV} \approx -6.3 \star 10^{29} u_s \quad \text{kg} \tag{4.4}$$

. The region that interests us in this study is of course the region where the newly inserted term B varies, meaning region II. Consequently for the next step the negative energy in region II, E_{II}, has to be calculated. With this in mind , first some values must be given to the following terms:

$$\begin{aligned} \alpha &= 10^{17} \\ \bar{\Delta} &= 10^{-15}\text{m} \\ \bar{R} &= 10^{-15}\text{m} \\ R &= 3 \star 10^{-15}\text{m} \end{aligned} \tag{4.5}$$

It is easily understood from these values that the surface of the **Warp Bubble** will have an area corresponding to a radius of $3 \star 10^{-15}$, while the inner diameter of the pocket is $\left(\bar{R} + \bar{\Delta}\right) \alpha = 200\text{m}$.
Furthermore for the calculation of the [WEC] an orthonormal frame will be chosen:

$$e_{\hat{0}} = \partial_t + u_s \partial_x \quad \text{and} \quad e_{\hat{i}} = \frac{1}{B} \partial_i \tag{4.6}$$

With the new orthonormal base in mind the [WEC] will take the form:

$$T_{\mu\nu} U^\mu U^\nu = T_{\hat{\mu}\hat{\nu}} U^{\hat{\mu}} U^{\hat{\nu}} = T^{\hat{\mu}\hat{\nu}} U_{\hat{\mu}} U_{\hat{\nu}} \tag{4.7}$$

For the next step some definitions will be used for convenience. First of all as it is already proven, when a observer is travelling on a time-like geodesic γ inside the metric's **Warp Bubble** its velocity is the same as an **Eulerian Observer** and the velocity is given in the form,$U_\mu = \alpha(-1, 0, 0, 0)$, where α was set to be 1. Secondly the [WEC] will be calculated in region II, when $t = 0$ and then r_s becomes $r_s = r = (x^2 + y^2 + z^2)^{\left(\frac{1}{2}\right)}$. Finally Equation(4.7) and what we are

going to calculate is given by:

$$T^{\hat{0}\hat{0}}U_{\hat{0}}U_{\hat{0}} = T^{\hat{0}\hat{0}}\alpha^2(-1)(-1) = \frac{1}{8\pi}G^{\hat{0}\hat{0}} \tag{4.8}$$

Since the *Van den Broeck* metric was given earlier in this chapter one can easily calculate the Einstein Tensor, G^{tt}, by simply putting the metric tensor in one of the many Mathematica programs made to produce the needed tensors and one then obtains:

$$G^{tt} = \left(\frac{1}{B^4}\left(\partial_r B\right)^2 - \frac{2}{B^3}\partial_r\partial_r B - \frac{4}{B^3}\partial_r B\frac{1}{r}\right) \tag{4.9}$$

However the value that was found for G^{tt} must now be converted to our new orthonormal metric and the method we will use in this case to find the Einstein Tensor in our newly set orthonormal bases will be the Fermi Normal Coordinates. The Fermi Normal Coordinates are being explained in **Appendix B** from where we will drain the equations needed for the conversion. Firstly we must declare our orthonormal base as it was set earlier by Equation(4.6):

$$\hat{e}{}^{a'}_{\mu} = \begin{cases} \hat{e}{}^{a'}_{0} = (1, u_s, 0, 0) \\ \hat{e}{}^{a'}_{1} = \left(0, \frac{1}{B}, 0, 0\right) \\ \hat{e}{}^{a'}_{2} = \left(0, 0, \frac{1}{B}, 0\right) \\ \hat{e}{}^{a'}_{3} = \left(0, 0, 0, \frac{1}{B}\right) \end{cases} \tag{4.10}$$

With the intention of finding $G^{\hat{0}\hat{0}}$ in mind, one can in addition use Equation(B.10) and come to the conclusion for the relation with which to obtain $G^{\hat{0}\hat{0}}$:

$$G^{\hat{0}\hat{0}} = \frac{\partial t}{\partial\hat{0}}\frac{\partial t}{\partial\hat{0}}G^{tt} + \frac{\partial t}{\partial\hat{0}}\frac{\partial x}{\partial\hat{0}}G^{tx} + \frac{\partial x}{\partial\hat{0}}\frac{\partial x}{\partial\hat{0}}G^{xx} + \frac{\partial y}{\partial\hat{0}}\frac{\partial y}{\partial\hat{0}}G^{yy} + \frac{\partial z}{\partial\hat{0}}\frac{\partial z}{\partial\hat{0}}G^{zz} \tag{4.11}$$

Likewise from Appendix B and Equation(B.9), one can derive the components of the previous relation, so as a result one obtains:

$$\frac{\partial t}{\partial \overset{\wedge}{0}} = e^{(t)}_{\overset{\wedge}{0}} \tag{4.12a}$$

$$\frac{\partial x}{\partial \overset{\wedge}{0}} = e^{(x)}_{\overset{\wedge}{0}} \tag{4.12b}$$

$$\frac{\partial y}{\partial \overset{\wedge}{0}} = e^{(y)}_{\overset{\wedge}{0}} \tag{4.12c}$$

$$\frac{\partial z}{\partial \overset{\wedge}{0}} = e^{(z)}_{\overset{\wedge}{0}} \tag{4.12d}$$

For the next step one can compare the components above with those given by Equation(4.10) so one can put values to these terms:

$$\begin{aligned} e^{(t)}_{\overset{\wedge}{0}} &= 1 \\ e^{(x)}_{\overset{\wedge}{0}} &= u_s \\ e^{(y)}_{\overset{\wedge}{0}} &= 0 \\ e^{(z)}_{\overset{\wedge}{0}} &= 0 \end{aligned} \tag{4.13}$$

Evidently the previous Equation(4.11) takes the much more simple and elegant form:

$$G^{\hat{0}\hat{0}} = G^{tt} + 2u_s G^{tx} + u_s^2 G^{xx} \tag{4.14}$$

But one should also consider that the calculation for the Einstein Tensor is happening at $t = 0$, where $r = r_s$ and $u_s = 0$. Finally the Einstein tensor in the given orthonormal base takes the form:

$$G^{\hat{0}\hat{0}} = G^{tt} = \left(\frac{1}{B^4} \left(\partial_r B \right)^2 - \frac{2}{B^3} \partial_r \partial_r B - \frac{4}{B^3} \partial_r B \frac{1}{r} \right) \tag{4.15}$$

Furthermore the tensor we need for the calculation of the [WEC],$T^{\hat{0}\hat{0}}$ has at last become:

$$T^{\hat{0}\hat{0}} = \frac{1}{8\pi} \left(\frac{1}{B^4} \left(\partial_r B \right)^2 - \frac{2}{B^3} \partial_r \partial_r B - \frac{4}{B^3} \partial_r B \frac{1}{r} \right) \tag{4.16}$$

For the purpose of calculating an exact value for $T^{\hat{0}\hat{0}}$ one must find a choice for the B function, for which $T^{\hat{0}\hat{0}}$ takes a minimum value.

As it can be easily derived from the Equation above, B function can't be a sine function or a low-order polynomial, cause if we try to calculate $T^{\hat{0}\hat{0}}$ we can ascertain that these type of functions have problems, in the sense that they have *Curvature Radii* smaller than the *Planck Length*. These problems occur due to the second derivative term, which also continues to the calculation of the *Riemann Tensor*. A solution to this particular problem is choosing a B function, which has a vanishing second order derivative. In this chapter we will choose at first an arbitrary B function but it will be shown later that this one is the best one possible.So B function is set as:

$$B = \alpha \left(- (n-1) w^n + n w^{n-1} \right) + 1 \tag{4.17}$$

with $w = \dfrac{\bar{R} + \bar{\Delta} - r}{\bar{\Delta}}$ and n particularly large. For the next step in the calculation of the [WEC] in region II a value for n will be chosen and it will be set $n = 80$. Using an easy Mathematica code, which we have built and after we substitute the values to $\alpha, \bar{R}, \bar{\Delta}$ we can easily discern that $T^{\hat{0}\hat{0}}$ is negative for:

$$\begin{aligned} 0 \leq & w \leq 0.981 \\ 1.019 \star 10^{-15} \leq & r \leq 2 \star 10^{-15} \end{aligned} \tag{4.18}$$

One can easily understand that $T^{\hat{0}\hat{0}}$ has a strong negative peak at:

$$\begin{aligned} &\text{for} \quad w = 0.349 \quad \text{we have} \quad T^{\hat{0}\hat{0}} = -4.9 \star 10^{32} \\ &\text{for} \quad r = 1.425 \star 10^{-15} \text{we have} \quad T^{\hat{0}\hat{0}} = -1.7529 \star 10^{32} \end{aligned} \tag{4.19}$$

One way to interpret the difference between the $T^{\hat{0}\hat{0}}$ is due to the specifications of the desktop we have used to run our Mathematica code because the numbers are very large but as one can easily ascertain we arrived at a good approximation. Below follows the plot of $T^{\hat{0}\hat{0}}(r)$ for different r:

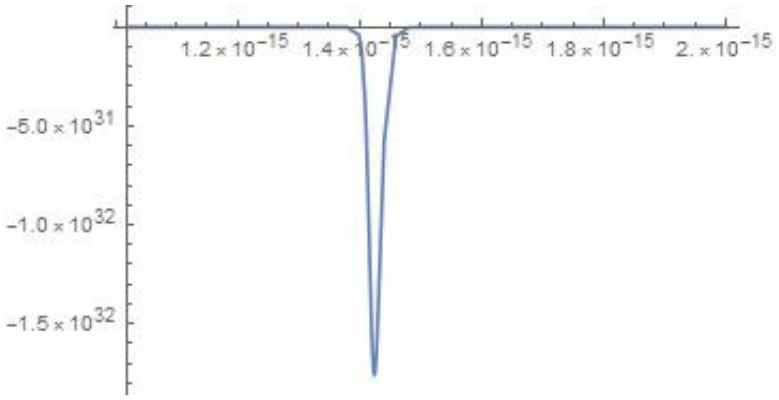

FIGURE 4.2: In this figure one can discern that $T^{\hat{0}\hat{0}}(r)$ is everywhere positive except for when r varies between $1.019 \star 10^{-15}$ and $2 \star 10^{-15}$ where not only is $T^{\hat{0}\hat{0}}(r)$ negative but it has a strong negative peak at $r = 1.425 \star 10^{-15}$

In the final analysis, the [WEC] definition will be used as it was given in the previous Chapter and particularly Equation(**??**). Then it will be integrated over the **Eulerian Observers** at t=0 and furthermore the limits of the integration will be the part of region II, where the energy density is negative as it was shown before:

$$\begin{aligned} E_{II,-} &= \int_{II,-} \sqrt{|\, g_s \,|} T_{\hat{\mu}\hat{\nu}} u^{\hat{\mu}} u^{\hat{\nu}} d^3x \\ &= 4\pi\bar{\Delta} \int_0^{0.981} dw\, (2-w)^2\, B(w)^3 T^{\hat{0}\hat{0}}(w) \\ &= -1.4 \star 10^{30}\text{kg} \end{aligned} \tag{4.20}$$

with $g_s = B^6$ as the determinant of the spatial metric on the surface $t = 0$. Also the $\dfrac{c^2}{G}$ term has been reinstated to obtain the answer in kg. The same integral in the positive part of the energy density tensor $T^{\hat{0}\hat{0}}$, meaning for $w > 0.981$ is :

$$E_{II,+} = 4.9 \star 10^{30} \tag{4.21}$$

For a final remark it is easily understood that the energy requirements of the **Warp Drive** have been reduced from the order of the mass in the entire visible universe to just in the order of few solar masses.

4.3 The new proposed Metric against the QI

Having considered the behaviour of the new proposed metric, when the [WEC] was applied to it, it is also reasonable to look at how the metric fares up against the [QI] as it was derived by Ford and Roman, and was used extensively in **Chapter 3**. The [QI] states as one can remember:

$$\frac{t_0}{\pi}\int_{-\infty}^{+\infty}\frac{< T_{\mu\nu}U^{\mu}U^{\nu} >}{t^2+t_0^2}dt \geq -\frac{3}{32\pi^2 t_0^4} \tag{4.22}$$

where t_0 is the sampling time. In the following calculation it will be taken as granted that the inequality should be valid in curved spacetimes, only if the sampling time is much smaller than the minimum curvature radius.
However the minimum curvature radius is determined by the largest component of the Riemann tensor. After these remarks for an easier calculation and without loss of the generality, a local coordinate transformation will be performed in region II,$x' = x - u_s t$, so that the metric becomes:

$$g_{\mu\nu} = diag\left(-1, B^2, B^2, B^2\right) \tag{4.23}$$

Additionally, if the calculation is limited to points on the line $y = z = 0$ and the largest component, in absolute value, is transformed to the orthonormal base in Equation(4.6) is:

$$R_{\hat{1}\hat{2}\hat{1}\hat{2}} = \frac{1}{B^4}\left(\partial_r B\right)^2 - \frac{1}{B^3}\partial_r^2 B - \frac{1}{B^3}\partial_r B\frac{1}{r} \tag{4.24}$$

The minimum curvature radius can be calculated where $R_{\hat{1}\hat{2}\hat{1}\hat{2}}$ is largest and as it was shown this means $w = 0.348$. This yields:

$$\begin{aligned} r_{c,min} &= \frac{1}{\sqrt{\mid R_{\hat{1}\hat{2}\hat{1}\hat{2}} \mid}} \\ &= \frac{\bar{\Delta}}{72.5} \\ &= 1.4 \star 10^{-34} \approx 10 \quad \text{Planck Lengths} \end{aligned} \tag{4.25}$$

For the sampling time it will be chosen:

$$t_0 =_{c,min}= 0.1 r_{c,min} \tag{4.26}$$

and concluding:

$$\hat{T^{00|}} \geq -\frac{3}{32\pi^2 t_0^4} \tag{4.27}$$

where after taking into account the hidden factors $\frac{c^2}{G}$ on the left and $\frac{\hbar}{c}$ on the right the left hand side is about $-6.6 \star 10^{93} \quad kgm^{-3}$ at the smallest value, as it was shown, while the right hand side is approximately $-9.2 \star 10^{94} \quad kgm^{-3}$. So for a final remark is was made clear that the [QI] is satisfied and the energy requirements of the original **Alcubierre Metric** can be reduced by a factor of approximately $10^{30} kg$.

Appendix A

Basics of Linearized Gravity

The following presentation of the linearized gravity is a sum of accumulated knowledge on the subject by Misner, Wheeler and Thorne. So for the first step, the metric of a weak gravitational field, linearized around a flat Minkowski space time can be written:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \text{with} h_{\mu\nu} \ll 1 \tag{A.1}$$

Fron now on, tensor indices will be raised and lowered with $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ instead of $g_{\mu\nu}$. Furthermore in the linear approximation $g^{\mu\nu}$ is given by $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. With the new metric in mind the connection coefficients can be written, when linearised as:

$$\begin{aligned} \Gamma^{\mu}_{\alpha\beta} &= \frac{1}{2}\eta^{\mu\nu}\left(h_{\alpha\nu,\beta} + h_{\beta\nu,\alpha} - h_{\beta,\nu}\right) \\ &= \frac{1}{2}\left(h^{\mu}{}_{\alpha,\beta} + h^{\mu}{}_{\beta,\alpha} - h_{\alpha\beta,}{}^{\mu}\right) \end{aligned} \tag{A.2}$$

Continuing with writing the terms of General Relativity in a linearized form, the *Ricci Tensor*, $R^{\alpha}_{\mu\nu}$, can be linearized and it can be given in terms of Christoffel symbols as follows:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} = \frac{1}{2}\left(h^{\alpha}{}_{\mu\nu,\alpha} + h^{\alpha}_{\nu,\mu\alpha} - h_{\mu\nu,}{}^{\alpha}\right) \tag{A.3}$$

where $h = h^{\alpha}_{\alpha} = \eta^{\alpha\beta}h_{\beta\alpha}$ is the trace of $(h_{\mu\nu}$. Furthermore, if someone wants to find the linearised *Ricci Scalar*:

$$R = g^{\mu\nu}R_{\mu\nu} \approx \eta^{\mu\nu}R_{\mu\nu} \tag{A.4}$$

Finally after understanding these new linearized terms one is able to proceed to the writing of the *linearized Einstein Tensor* in the following form:

$$G_{\mu\nu} = \frac{1}{2}\left(h_{\mu\alpha,\nu}{}^{\alpha} + h_{\nu\alpha,\mu}{}^{\alpha} - h_{\mu\nu,\alpha}{}^{\alpha} - h_{,\mu\nu} - \eta_{\mu\nu}\left(h_{,\alpha\beta}{}^{\alpha\beta} - h_{,\beta}{}^{\beta}\right)\right) \tag{A.5}$$

Thus the *linearized Einstein Tensor* was defined, in terms of $(h_{\mu\nu})$. However one can conclude to a more easily handled form of the same tensor,, if one takes the following steps. Initially the trace reverse of $(h_{\mu\nu})$ must be taken from the equation:

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h \tag{A.6}$$

where $\bar{h} = \bar{h}^{\alpha}_{\alpha} = -h$. So the *linearized Einstein Tensor* can also be written in terms of $\bar{h}_{\alpha_\beta}$:

$$G_{\alpha\beta} = -\frac{1}{2}\left(\bar{h}_{\alpha\beta,\mu}{}^{\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu,}{}^{\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{\mu} - \bar{h}_{\beta\mu,\alpha}{}^{\mu} + O(\bar{h^2}_{\alpha\beta})\right) \tag{A.7}$$

Last but not least, another useful equation derived from the linearised gravity theory is the *linearized Einstein Equation*:

$$-\bar{h}_{\alpha\beta,\mu}{}^{\mu} - \eta_{\alpha\beta}\bar{h}_{\mu\nu,}{}^{\mu\nu} + \bar{h}_{\alpha\mu,\beta}{}^{\mu} + \bar{h}_{\beta\mu,\alpha}{}^{\mu} = 16\pi T_{\alpha\beta} \tag{A.8}$$

Appendix B

Fermi Normal Coordinates

In this Appendix the main focus will be to obtain the *Fermi Normal Coordinates*,as they were being described by Poisson, and then use them to find a way to convert the components of an arbitrary metric to components in relation to the metric described by the *Fermi Normal Coordinates* .

Initially the local flatness theorem must be stated and explored. This theorem states that for a given point P in space time, it is always possible to find a coordinate system x^{α} such that:

$$g_{\alpha'\beta'}(P) = \eta_{\alpha'\beta'} \quad \text{and} \quad \Gamma^{\alpha'}_{\beta'\gamma'}(P) = 0 \tag{B.1}$$

where $\eta_{\alpha'\beta'} = diag(-1, 1, 1, 1)$ is the Minkowski space time. Such a coordinate system that allows these relations is called a local Lorentz frame. The physical interpretation of the local flatness theorem is that a free falling observer can see no effect of gravity in their immediate vicinity, which corresponds to Einstein's equivalence principle.

However there was no mention of how one is able to construct a coordinate system with these qualities. For this reason a more geometric proof of the theorem will be provided and it will be extended from one point P to a geodesic γ, which in turn for convenience will be considered time-like. For this analysis first step it will be shown that coordinates of the form $x^{\alpha} = (t, x^{a})$ can be introduced such that near

γ the metric becomes:

$$\begin{aligned} g_{tt} &= -1 - R_{tatb}(t)x^a x^b + O(x^3) \\ g_{t\alpha} &= -\frac{2}{3}R_{tbac}(t)x^b x^c + O(x^3) \\ g_{ab} &= \delta_{ab} - \frac{1}{3}R_{acbd}(t)x^c x^d + O(x^3) \end{aligned} \tag{B.2}$$

These coordinates are known as *Fermi Normal Coordinates*, where t is the proper time along γ, on which spatial components x^α are zero. Another interesting aspect of these new coordinates is that they enforce Equation(B.4).
For the next step the connection between the *Fermi Normal Coordinates*, symbolised by $x^\alpha = (t, x^a)$, and some arbitrary coordinate system, symbolised by $x^{a'}$, will be found. In consequence, a space time will be described in terms of the arbitrary metric and it will be in the form of $g_{\alpha'\beta'}$. In this metric a time like geodesic γ is considered, where its tangent vector is $u^{\alpha'}$ and its proper time is denoted by t. A point O will be defined on γ at which its proper time will be set $t = 0$. Eventually at this point an orthonormal basis $\hat{e}{}_\mu^{\alpha'}$, where μ denotes the four basis vectors, will be erected at 0 and then $\hat{e}{}_t^{\alpha'}$ will be identified with the tangent vector $u^{\alpha'}$ at O. From this a new basis can be constructed everywhere on γ by parallel transporting $\hat{e}{}_\mu^{\alpha'}$ away from O. These new vectors must satisfy:

$$\hat{e}{}_{\mu\beta'}^{\alpha'} u^{\beta'} = 0 \tag{B.3a}$$

$$\hat{e}{}_t^{\alpha'} = u^{\alpha'} \tag{B.3b}$$

$$g_{\alpha'\beta'} \hat{e}{}_\mu^{\alpha'} \hat{e}{}_\nu^{\beta'} = \eta_{\mu\nu} \tag{B.3c}$$

Now a spacelike geodesic β can be considered at a point P on γ where $t = t_p$. On this new geodesic β a tangent vector can be set with $w^{\alpha'}$ and its proper distance will be denoted with s. It is also assumed that at point P, $w^{\alpha'}$ is orthogonal to $u^{\alpha'}$ so that :

$$w^{\alpha'}|_\gamma = \Omega^\alpha \hat{e}{}_\alpha^{\alpha'} \tag{B.4}$$

It must be noted at this point that to ensure that $w^{\alpha'}$ is normalized the expansion coefficients must satisfy $\delta_{ab}\Omega^a\Omega^b = 1$. However one can come to the conclusion that with different coefficients other geodesics can be constructed with are also orthonormal to γ at point P. So from now the entire family of these geodesics

will be identified by $\beta(t_p, \Omega^a)$. The Fermi coordinates can now be constructed as follows on point P. Initially point Q is assigned the coordinates:

$$x^0 = t_p \quad and \quad x^\alpha = \Omega^\alpha_Q s_Q \tag{B.5}$$

with s_Q as the proper distance between points Q and P. Generically therefore, $x^\alpha = (t, \Omega^\alpha_s)$ and the next step is to figure out how these coordinates are related to $x^{\alpha'}$, the original system. If $s = 0$ is substituted in Equation(B.5) the description of

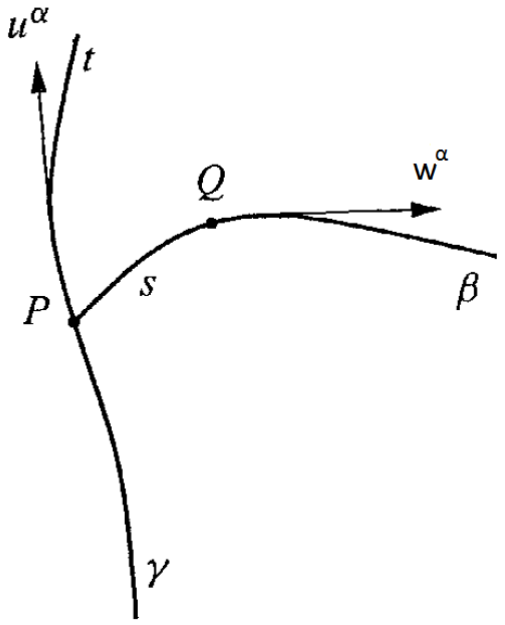

FIGURE B.1: This figure shows geodesic β, which is orthonormal on geodesic γ , points P and Q and also the tangent vectors u^α and w^α on these two points respectively

the timelike γ is recovered in terms of proper time t; the parameters Ω^α are then irrelevant. From the figure below the tangent to the family of geodesics $\beta(t, \Omega^\alpha)$ can be denoted as:

$$w^{\alpha'} = \left(\frac{\partial x^{\alpha'}}{\partial s}\right)_{t,\Omega^\alpha} \tag{B.6}$$

One can easily come to the conclusion that this newly appointed vector is a solution to the geodesic equation subjected to the initial condition $w^{\alpha'} \mid_{s=0} = \Omega^\alpha \hat{e}^{\alpha'}_\alpha$. However the geodesic is invariant under a rescale of the affine parameter $s \to \frac{s}{c}$. Under this rescaling one also takes the following changes such as, $w^{\alpha'} \to c w^{\alpha'}$ and $\Omega^\alpha \to c\Omega^\alpha$. From this point onward the new identity can be established

$x^{\alpha'}(t, \Omega^\alpha, s) \to x^{\alpha'}\left(t, c\Omega^\alpha, \frac{s}{c}\right)$ and as a special case:

$$x^{\alpha'}(t, \Omega^\alpha, s) = x^{\alpha'}(t, \Omega^\alpha_s, 1) \equiv x^{\alpha'}(x^\alpha) \tag{B.7}$$

Finally one arrives at the conclusion from using the above relation and Equation(B.5) that this relation is the transformation between $x^{\alpha'}$ and the *Fermi Normal Coordinates*. Closing this appendix the relations for the construction of the *Fermi Normal Coordinates* can be discovered by:

$$\Omega^\alpha \hat{e}{}^{\alpha'}_\alpha = w^{\alpha'} \mid_\gamma = \left(\frac{\partial x^{\alpha'}}{\partial s}\right) \mid_{s=0} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \mid_{s=0} \Omega^\alpha \tag{B.8}$$

which shows:

$$\frac{\partial x^{\alpha'}}{\partial x^\alpha} \mid_\gamma = \hat{e}{}^{\alpha'}_\alpha \tag{B.9}$$

Last but not least, another useful relation coming from the *Fermi Normal Coordinates* is the one showing how the components of the metric written in the *Fermi Normal Coordinates* are related to the old components:

$$g_{\alpha\beta} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^{\beta'}}{\partial x^\beta} g_{\alpha'\beta'} \tag{B.10}$$

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